Lecture 8

CSE 431
Intro to Theory of Computation

Last time:

$A_{TM} = \{\langle M, w\rangle : M \text{ is a TM that accepts } w\}$

Thm A_{TM} is Turing-recognizable

Proof universal TM U ∎

Thm (Cantor) $\mathcal{P}(\mathbb{N})$ is not countable

Proof Assume by contradiction that $P(\mathbb{N})$ is countable.
$\Rightarrow$ there is a listing $S_0, S_1, \ldots$ of all subsets of $\mathbb{N}$.
Show that this list must miss some set $D \subseteq \mathbb{N}$ "flipped diagonal" set ∎

Thm Σ^* is countable

Proof dovetailing: list by length, then break ties within each length by integer value
$\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$ ∎

Thm $\{M : M \text{ is a TM}\}$ is countable

Pf $\{M : M \text{ is a TM}\}$
same size
$\{\langle M\rangle : M \text{ is a TM}\} \subseteq \Sigma^*$ alphabet encoding TMs ∎

Claim $\mathcal{P}(\Sigma^*)$ is not countable

set of all possible languages over Σ

Proof Choose any $a \in \Sigma$

$$\mathcal{P}(\mathbb{N}) \xleftrightarrow{\text{same size}} \mathcal{P}(\{a^n : n \geq 0\}) \subseteq \mathcal{P}(\Sigma^*)$$ ∎

Cor There is some language that is not Turing-recognizable

Proof $|\{\text{TMs}\}| < |\{\text{languages}\}|$ ∎

More precise info

Thm A_{TM} is not decidable

Proof Suppose: A_{TM} is decidable (*) by some TM H.

On input $\langle M, w \rangle$:
- H must accept if M accepts
- H must reject if M does not accept (M rejects or runs forever)

We give two proofs:

Proof 1 We consider a table of the behavior of H on a variety of inputs $\langle M, w \rangle$

Focus on strings w that are codes of TMs

Can list all TMs since set of TMs is countable

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	...	
M_1	1	0	1	1	0	?
M_2	0	1	0	0	1	0
M_3	1	1	0	1	0	0
M_4	0	1	0	0	0	1 ...
M_5	1 ...	0	1	1	0	...
⋮						
D	0	0	1	1	1	

(i,j) entry $= \begin{cases} 1 & \text{iff } M_i \text{ accepts } \langle M_j\rangle \text{ iff } H \text{ accepts } \langle M_i, \langle M_j\rangle\rangle \\ 0 & \text{iff } M_i \text{ does not accept } \langle M_j\rangle \text{ iff } H \text{ rejects } \langle M_i, \langle M_j\rangle\rangle \end{cases}$

Given the TM H, this motivates defining a TM D ~~for~~ the flipped diagonal language. as follows:

D: On input $\langle M\rangle$:
- Let $w = \langle M\rangle$
- Run H on input $\langle M, w\rangle$: ($\langle M, \langle M\rangle\rangle$)
  - if H accepts then reject
  - if H rejects then accept

For any i:
Since D behaves differently from M_i on input $\langle M_i\rangle$, $D \neq M_i$

However, by construction the listing of TMs
$M_1, M_2, \ldots$
was a complete list which is a
contradiction to our assumption
$\circledast$

$\therefore$ A_{TM} is not decidable $\square$

Proof 2: In this version we ignore the table
and just use the definition
of D.
Since H is a decider, D is a decider
We consider the following question:
Does D accept $\langle D \rangle$?

Now D accepts $\langle D \rangle$
$\Rightarrow$ H accepts $\langle D, \langle D \rangle \rangle$ by defn of H
$\Rightarrow$ D rejects $\langle D \rangle$ by defn of D
contradiction

But also D rejects $\langle D \rangle$
$\Rightarrow$ H rejects $\langle D, \langle D \rangle \rangle$ by defn of H
$\Rightarrow$ D accepts $\langle D \rangle$ by defn of D
contradiction

Either way we get a contradiction so the only possibility is that D doesn't exist which implies that H doesn't exist

Notes:

- In the first proof, unlike the case of $\mathcal{P}(\mathbb{N})$, the table does exist. The issue is assuming that there is decider H that corresponds to it.
- U is like H except that
  - U exists
  - U can run forever (in which case H must reject)

  If we replace H with U in D we don't get a contradiction because we can't "flip" running forever.

This is the essence of Turing's proof of the Undecidability of the Halting Problem

though strictly speaking A_{TM} is different

We can use this to find an explicit language that is not Turing-recognizable

Thm A language A is decidable $\iff$ A and $\overline{A}$ are both Turing-recognizable

Proof ($\Rightarrow$) A decidable $\Rightarrow$ $\bar{A}$ decidable
(swap q_{acc}, q_{rej})
$\Downarrow$ $\Downarrow$
A Turing-recognizable $\bar{A}$ Turing-recognizable

($\Leftarrow$) Suppose A and $\bar{A}$ are Turing-recognizable by TMs M_A and $M_{\bar{A}}$:

Decider M for A:
On input w:
Copy w to a second tape

Run M_A and $M_{\bar{A}}$ alternately one step at a time on each tape

One of M_A or $M_{\bar{A}}$ will halt and accept first

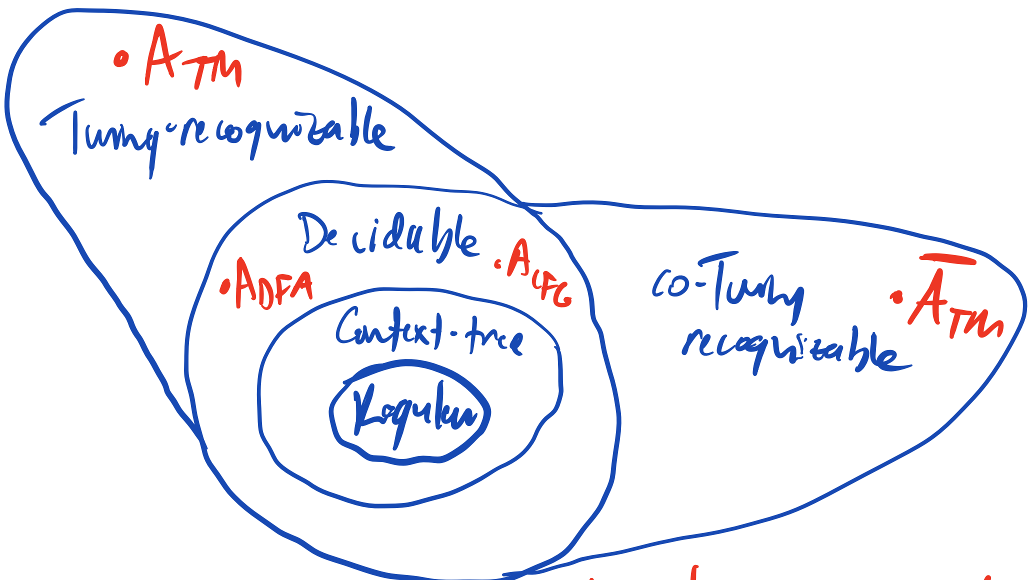
- if M_A accepts then accept
- if $M_{\bar{A}}$ accepts then reject ∎

Cor $\overline{A_{TM}}$ is not Turing recognizable

Proof Since A_{TM} is T-rec, if $\overline{A_{TM}}$ were T-rec the A_{TM} would be decidable which it isn't ∎

Def^n A is <u>co-Turing recognizable</u>
iff $\overline{A}$ is Turing-recognizable

We have the following picture of the space of languages over Σ.



We can now show many other languages undecidable

Def^n $HALT_{TM} = \{\langle M, w\rangle : \text{TM } M \text{ halts on input } w\}$

This is the Halting Problem Turing considered. We now show it also is undecidable.

Thm $HALT_{TM}$ is undecidable

Proof Today we give one proof. Next time we give another

Suppose $HALT_{TM}$ were decidable with

decider R

Idea: Show that if we had R then we could get a decider for A_{TM} as follows (which is impossible)

TM S: On input $\langle M,w\rangle$:
Run R on input $\langle M,w\rangle$:
If R rejects then reject
If R accepts then Run U on input $\langle M,w\rangle$

will always halt since R accepts

S would be a decider for A_{TM} which can't exist so R doesn't exist $\boxtimes$

$E_{TM} = \{\langle M\rangle \mid M \text{ is a TM with } L(M)=\emptyset\}$

Thm E_{TM} is undecidable

Proof Suppose we had a decider E for E_{TM}

Idea: we show that with E we could build a decider for A_{TM} (impossible)

TM F: On input $\langle M,w\rangle$
- Modify M to erase its input and replace it with w on its tape and then behave as M does from there.
- Call new TM M_w and code $\langle M_w\rangle$.

- Run E on input $\langle M_w \rangle$
- If E accepts then reject

  If E rejects then accept

Observe that $L(M_w)$ is either Σ^* (if M accepts w)

or ϕ (if M doesn't accept w)

M accepts w $\Rightarrow$ $L(M_w) = \Sigma^*$

$\Rightarrow$ E rejects $\langle M_w \rangle$

$\Rightarrow$ F accepts $\langle M, w \rangle$

M doesn't accept w $\Rightarrow$ $L(M_w) = \phi$

$\Rightarrow$ E accepts $\langle M_w \rangle$

$\Rightarrow$ F accepts $\langle M, w \rangle$

i.e. F decides A_{TM} contradiction